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LETTER TO THE EDITOR

The modified resolvent for the one-dimensional Schrödinger operator with a reflectionless potential and Green functions in multidimensions

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Abstract. The representation of the resolvent of the one-dimensional Schrödinger operator with a reflectionless potential does not contain the integral term. It is obtained via Jost function simple poles decomposition within the formula for the Green function. As a corollary we have a new expression for the reflectionless potential with a free parameter. The results allow the Green functions for a wide class of multidimensional differential equations with a coefficient proportional to the potential to be obtained. Examples for diffusion and wave equations are given.

The integral term in the kernel of the resolvent for the Schrödinger operator on one x axis

$$L = -\mathrm{d}^2/\mathrm{d}x^2 + u(x) \qquad \int (1+|x|)|u(x)|\,\mathrm{d}x < \infty \tag{1}$$

by the eigenfunctions of the discrete and continuous spectra is simplified in the case of the reflectionless potential $u(x)$ but does not vanish. In this letter we present an explicit representation for the kernel that contains only the sum over the discrete spectrum points without the integral term and demonstrate the use of such formalism. The method of construction goes up to the factorization technique, see, for example the review [1].

Let the potential $u(x)$ in (1) be the reflectionless one and have n bounded states. Denoting $\lambda_m = -b_m^2$, $b_m > 0$, $m = 1, \ldots, n$ for the points of the discrete spectrum and the corresponding eigenfunctions $\psi_m(x)$, that we normalize as

$$\lim_{x\to\infty} \psi_m(x)\mathrm{e}^{b_m x} = 1.$$

Let the Jost function $\psi(x,k)$ be fixed by the condition

$$\lim_{x\to\infty} \psi(x,k)\mathrm{e}^{-\mathrm{i}kx} = 1.$$

It is clear that $\psi_m(x) = \psi(x, \mathrm{i}b_m)$ and the Jost function in the case of n-level reflectionless potential allows the representation [2]

$$\psi(x,k) = \mathrm{e}^{\mathrm{i}kx} R(x,k) \qquad R(x,k) = P_n(x,k) \Big/ \prod_{m=1}^{n} (k + \mathrm{i}b_m) \tag{2}$$

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where $P_n(x,k)$ is a polynomial in k of nth power with the senior term k^n so that

$$\lim_{k\to\infty} R(x,k) = 1.$$

The decomposition of $R(x,k)$ in the simple decimals has the form [2]

$$R(x,k) = 1 - \mathrm{i}\sum_{m=1}^{n} \rho_m \psi_m(x) \mathrm{e}^{-b_m x}/(k + \mathrm{i}b_m) \tag{3}$$

where $\rho_m = [\int_{-\infty}^{\infty} \psi_m^2(x)\,\mathrm{d}x]^{-1}$. The simplest derivation of the representation (2) may be obtained by the Darboux transformation technique [3]. The resolvent kernel $G(x, x_0, k)$ of the operator L satisfies the equation

$$(L - k^2)G(x, x_0, k) = \delta(x - x_0) \tag{4}$$

and is expressed at $\operatorname{Im} k > 0$ by means of the Jost function $\psi(x,k)$ in the following way

$$2\mathrm{i}kG(x, x_0, k) = \begin{cases} \psi(x, -k)\psi(x_0, k) & x < x_0 \\ \psi(x, k)\psi(x_0, -k) & x > x_0. \end{cases} \tag{5}$$

From (2)–(5) it follows that

$$G(x, x_0, k) = -\mathrm{e}^{\mathrm{i}k|x-x_0|} S(x, x_0, k)/2\mathrm{i}k \tag{6}$$

where $S(x, x_0, k)$ is a rational function of k, that is symmetrical in x, x_0 and has simple poles at the points $k = \pm\mathrm{i}b_m$, $m = 1, \ldots, n$, and $\lim_{k\to\infty} S(x, x_0, k) = 1$. Using (3) we get

$$\operatorname*{res}_{k=\pm\mathrm{i}b_m} S(x, x_0, k) = \pm\mathrm{i}\rho_m \psi_m(x)\psi_m(x_0)\exp(b_m|x - x_0|).$$

Now we can obtain the resulting formula for the kernel

$$G(x, x_0) = -\exp(\mathrm{i}k|x - x_0|)/2\mathrm{i}k - \sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0)[\exp(\mathrm{i}(k - \mathrm{i}b_m)|x - x_0|)/(k - \mathrm{i}b_m)$$
$$- \exp(\mathrm{i}(k + \mathrm{i}b_m)|x - x_0|)/(k + \mathrm{i}b_m)]/2k. \tag{7}$$

Let us give two consequences of the representation (7). After the straight substitution of (7) in equation (4) we see that the pole terms disappear and the remaining terms give the identity

$$u(x) = -4\mathrm{d}/\mathrm{d}x \sum_{m=1}^{n} \rho_m \sinh[b_m(x - x_0)]\psi_m(x)\psi_m(x_0) \tag{8}$$

which in the particular case $x = x_0$ gives the well known representation of a reflectionless potential by the squares of the eigenfunctions. If one introduces as in (1) the function $w(x) = \int u(\xi)\,\mathrm{d}\xi$ then the symmetric relation is obtained by integration of (8) with respect to x:

$$w(x) - w(y) = 4\sum_{m=1}^{n} p_m \sinh[b_m(x - y)]\psi_m(x)\psi_m(y).$$

Using the derived relations (7), (8) one can find Green functions for a wide class of multidimensional differential equations. Let L_0 be a linear differential operator in a new variable y with constant coefficients and let $E(x,y)$ be the fundamental function of the operator $L_0 - \partial^2/\partial x^2$, i.e.

$$(L_0 - \partial^2/\partial x^2)E(x, y) = \delta(x, y) \tag{9}$$

then for the Green function for the operator $L + L_0$ such that

$$(L_0 + L)G(x, y, x_0, y_0) = \delta(x - x_0, y - y_0) \tag{10}$$

the following expression is valid:

$$G(x,y,x_0,y_0) = E(x-x_0, y-y_0) + \sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0) E_m(x-x_0, y-y_0) \tag{11}$$

where $E_m(x, y)$ is some appropriate solution of the equation

$$\partial E_m(x,y)/\partial x = -2\sinh(b_m x)E(x,y). \tag{12}$$

Formula (11) is tested by the straight substitution in equation (11) using the equalities (7), (9), (10), and (12).

Example 1. The Green function of the operator $\partial/\partial t + \nu L$, $\nu > 0$ is

$$G(x,t,x_0,t_0) = \theta(t-t_0)\Bigg\{ \exp[-(x-x_0)^2/4\nu(t-t_0)]/2\sqrt{\pi}\nu(t-t_0)$$
$$+\sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0)\exp(-\nu b_m^2(t-t_0))[\mathrm{Erf}((x-x_0+2\nu b_m(t-t_0))$$
$$/2\sqrt{\nu}(t-t_0)) - \mathrm{Erf}(x-x_0-2\nu b_m(t-t_0)/2\sqrt{\nu}(t-t_0))]\Bigg\}/2.$$

Example 2. The Green function of the operator $c^{-2}\partial^2/\partial t^2 + L$ is

$$G(x,t,x_0,t_0) = c\theta(c(t-t_0)-|x-x_0|)\Bigg\{1/2 + \sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0)[\cosh(cb_m(t-t_0))$$
$$-\cosh(b_m(x-x_0))]/b_m\Bigg\}.$$

Both representations are obtained by means of well known fundamental solutions for one-dimensional thermoconductivity and wave operators.

The given representations for Green functions may be useful for the description of the mass and heat diffusion as well as for the wave propagation in inhomogeneous medium. The stratification of the considered form may be induced by soliton propagation. The technique is obviously applicable to other operators with factorizable L with the potentials that go to constants at infinities [4] (for factorization in multidimensions see [5]) but we restrict ourselves in this letter assuming that the most interesting point in our work is connected with representation (8) and generalization of higher dimension. We also hope for applications of our results in the resolvent approach in nonlinear theory [6] and in quantum models in one-loop approximation [3, 7] where exact solutions enter the temperature Green function equation [8].

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